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AN APPROXIMATION IN THE PROBLEM OF CONTROLLING THE SHAPE OF THE REGION FOR A PARABOLIC SYSTEM*

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A study is presented of the problem of optimal control of the shape of the region for the heat-conduction equation. It is proposed to approximate the problem by a penalty method applied to the shape of the region. The approximation leads to the investigation of a standard control problem for a bilinear parabolic system in a fixed region.

In applications one quite frequently encounters problems requiring the solution of the heat-conduction equation for regions whose shapes vary with time /1-4/.

Such problems arise when one is studying energy or mass transfer problems related to changes in the state of aggregation of matter /5, 6/, in problems of dam theory, soil mechanics, the temperature distribution in oil layers, and in filtration problems. The solution of diffusion problems for regions with moving boundaries is basic for the theory of zone refining of materials /3/. From the mathematical point of view, boundary-value problems of heat conduction in a region with a moving boundary are essentially different from classical heat-conduction problems. Since the geometrical dimensions of the region depend on time, general problems of this type cannot be dealt with by the classical methods normally used in mathematical physics.

Problems of this sort for elliptic control problems were formulated in the survey article /7/. A thorough treatment of these problems may be found in /8/.

The major complication to be overcome in studying control problems in time-variable regions is that one must deal with state functions $y(t, x)$ which are defined at different times t in different regions in x space. To that end, this paper proposes a method based on penalties, which depend on the shape of the region, which enables us to reduce a problem with a variable region to a problem defined in a preassigned fixed set. The latter differs in structure from the original region only in the presence of an additional penalty term, which contains all the information on the shape of the region and is an analogue of a bilinear control system depending on a large parameter.

The present paper is devoted to an investigation of the properties of this new approximating system.

1. *Statement of the problem.* Consider a system whose state is described by the following equations:

$$y^{\cdot}(t, x) = y''(t, x), \quad 0 < t < T, \quad 0 < x < u(t) \tag{1.1}$$

$$y(0, x) = \varphi(x), \quad 0 \leqslant x \leqslant u_0 \tag{1.2}$$

$$y(t, 0) = 0, \; y(t, u(t)) = 0, \quad 0 \leqslant t \leqslant T \tag{1.3}$$

$$u^{\cdot}(t) = v(t), \text{ a.e. } t \in [0, T], \quad v(t) \in [0, V] \tag{1.4}$$

$$u(0) = u_0 \tag{1.5}$$

Here $y(t, x)$ is the state of the control system at time t, which is a measurable function of x in the interval $[0, u(t)]$, and $\varphi(x)$ is the initial distribution of y. The function $u(t)$ is absolutely continuous and satisfies Eq.(1.4) for almost all t; $v(t)$ is a Lebesgue-measurable function in $[0, V]$, which plays the role of control and defines the "shape" of a non-cylindrical region $Q(u) = \{(t, x) \mid t \in (0, T), 0 < x < u(t)\}$ in which the arguments t and x are allowed to vary. Dots denote differentiation with respect to time and primes denote differentiation with respect to x. Thus, the state of the system at any time $t \in [0, T]$ is a function $y(t, x)$, defined for $x \in [0, u(t)]$.

On the states of this system, considered at time T, we are given a functional which plays the part of a cost criterion:

$$J(v(\cdot)) = G(y(T, \cdot)) \tag{1.6}$$

Our problem is to determine a control $v^\circ(\cdot)$ which minimizes this functional: $J(v^\circ(\cdot)) =$

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$\inf J(v(\cdot))$ over all admissible controls $v(\cdot)$. (The method proposed below to simulate problems of controlling the shape of a region can be extended to optimization problems with other cost criteria; see, e.g., Sect.4 below).

To solve this problem, we propose the following mathematical model. Choose a number X so that, for all $v(t) \in [0, V]$, $u(t) \in [0, X]$, and, fixing a region $D = (0, T) \times (0, X)$, consider the following model problem, which depends on a parameter $\varepsilon > 0$:

$$y_\varepsilon^{\cdot}(t, x) - y_\varepsilon''(t, x) + \varepsilon^{-1} U(t, x)\, y_\varepsilon(t, x) = 0,\ (t, x) \in D \tag{1.7}$$

$$y_\varepsilon(0, x) = \psi(x) = \begin{cases} 0, & x \in (u_0, X] \\ \varphi(x), & x \in [0, u_0] \end{cases}$$

$$y_\varepsilon(t, x) = 0, \quad (t, x) \in \Sigma = [0, T] \times (\{0\} \cup \{X\})$$

$$u^{\cdot}(t) = v(t),\ \text{a.e.}\ t \in [0, T],\ u(0) = u_0 \tag{1.8}$$

$$U(t, x) = \begin{cases} 1, & x \in [u(t), X] \\ 0, & x \in [0, u(t)) \end{cases},\ t \in [0, T]$$

The last term in Eq.(1.7) plays the role of a penalty term. The state of the new control system $y_\varepsilon(t, x)$ at any time t is a function defined for $x \in [0, X]$. The variation with time of the shape of the region is described by the function $U(t, x)$. In structure, Eqs.(1.7) and (1.8) constitute a bilinear control system of parabolic type, depending on the approximation parameter $\varepsilon > 0$. It is natural to expect that as $\varepsilon \to 0$ the solutions of this system will approximate the solutions of system (1.1)-(1.5) in a suitable function space.

In the sequel we shall consider different cost criteria G, defined on various function spaces. An example is the functional

$$G(z(\cdot)) = \int_0^X [z(x) - g(x)]^2\, dx \tag{1.9}$$

where $g(x)$ is a given function in $L_2(0, X)$ (see also Sect.4).

Control systems of type (1.7)-(1.9) have been thoroughly investigated in the literature on optimal control in distributed systems /4, 7/. Solutions which depend on the parameter ε exist and there is an abundance of approximate methods for determining them.

The main result of this paper is the statement that as $\varepsilon \to 0$ the optimal solutions of problem (1.7)-(1.9) converge to an optimal solution of problem (1.1)-(1.6).

2. *Mathematical formalization.* We shall use the following notation. $H^1(\Omega)$ is the Sobolev space of first order on the set Ω /9-11/; $f|_S$ denotes the restriction of the function f (the trace, if it exists) to the set S; $H_0^1(\Omega) = \{f \in H^1(\Omega) \mid f|_\Gamma = 0,\ \Gamma = \partial\Omega\}$; $\operatorname{supp} f$ is the support of a function (distribution) f; $\Sigma(u) = [0, T] \times \{0\} \cup \{(t, x) \mid t \in [0, T],\ x = u(t)\}$.

Definition. A solution of problem (1.1)-(1.5) is any function y satisfying the conditions

$$y \in H^1(Q(u)), \quad y|_{\Sigma(u)} = 0, \quad y|_{t=0} = \varphi$$

and the integral identity

$$\iint_{Q(u)} \{y^{\cdot}\varkappa + y'\varkappa'\}\, dx dt = 0,\ \forall \varkappa \in \Phi(u) =$$
$$\{\varkappa \in C^\infty(\overline{Q(u)}) \mid \forall t \in [0, T]\ \operatorname{supp} \varkappa(t, \cdot) \subset (0, u(t)),\ \varkappa(T, x) \equiv 0\}$$

A generalized solution of the approximating system will be understood in the sense of the theory of distributions - an element of the space $L_2(0, T; H_0^1(0, X)) \cap H^1(0, T; L_2(0, X)) \cap L_\infty(0, T; L_2(0, X))$ /9-11/.

Standard methods /9-11/, coupled with the use of a priori estimates in energy norms, yield a proof of the following theorem.

Theorem 1. Let $\varphi \in H_0^1(0, u_0)$. Then a unique solution of problem (1.1)-(1.5) exists.

3. *Main results.* Let $y(t, x; u(\cdot))$ denote a solution of system (1.1)-(1.5), defined as zero on the set $D \setminus \overline{Q(u)}$ and corresponding to a specific function $u(\cdot)$ determining the shape of the region. Let $y_\varepsilon(t, x; u(\cdot))$ be the corresponding solution of system (1.7), (1.8).

Theorem 2 (uniform approximation with respect to $u(\cdot)$. As $\varepsilon \to 0$ $y_\varepsilon(T, \cdot; u(\cdot)) \to y(T, \cdot; u(\cdot))$ weakly in $L_2(0, X)$ and uniformly on the set of all admissible controls $\{u(\cdot)\}$.

Remarks. 1. Because of the restrictions imposed on the parameters of the system, the admissible controls $u(\cdot)$ are monotone increasing functions. This enables one, using standard methods /10/, to establish energy estimates for the derivatives $y_\varepsilon^{\cdot}$ in the metric of the space $L_2(0, T; L_2(0, X))$ which are uniform in $\{u(\cdot)\}$ and independent of $\varepsilon > 0$; in the final analysis, this is what guarantees the desired uniform approximation property.

2. Using a probability representation for the solution of system (1.7), (1.8), /12, 13/, one can prove a stronger assertion, according to which y_ε converges to y in the metric $L_2(0, X)$ uniformly on a larger class of admissible controls $\{u(\cdot)\}$ (not necessarily monotone).

3. Since the convergence of $y_\varepsilon(T, \cdot; u(\cdot))$ to $y(T, \cdot; u(\cdot))$ is uniform in $\{u(\cdot)\}$, one can consider (1.7), (1.8) as an approximation to system (1.1)-(1.5). An optimal control $v_\varepsilon^\circ(\cdot)$ solving the optimization problem (1.7)-(1.9) exists /11, 14/ and, since the set $\{v(\cdot)\}$ is compact in the weak topology of $L_2(0, T)$, it converges as $\varepsilon \to 0$ to a control $v^\circ(\cdot)$ which is a solution of problem (1.1)-(1.6). Thus, the control problem for system (1.1)-(1.5) can be approximated by a control problem for the standard control system (1.7), (1.8).

4. The approximating system differs from the system under consideration only in the presence of the additional penalty term.

5. If the regions $Q(u)(\tau) = Q(u) \cap \{t = \tau\}$ vary monotonically with time (in the sense of set inclusion), the above results admit of a natural generalization to more general equations of parabolic type, in which x may take values in several dimensions.

4. As an example of the approximation method proposed above, we will consider its application to the single-phase Stefan problem.

The classical model of the one-dimensional frontal Stefan problem is described by the following equations /5, 6/:

$$\begin{gathered} y^{\cdot}(t, x) = y''(t, x), \quad t > 0, \quad 0 < x < u(t) \\ y(0, x) = \varphi_0(x), \quad 0 \leqslant x \leqslant u_0; \quad y(t, 0) = 0, \quad y(t, u(t)) = 0 \\ y'(t, u(t)) = -ku^{\cdot}(t), \quad t > 0; \quad u(0) = u_0 \end{gathered} \tag{4.1}$$

Here $y(t, x)$ is the temperature of the "liquid" phase at a point x at time t. The function $\varphi_0(\cdot)$ describes the initial distribution of temperature in the "liquid" phase, and $u(t)$ describes the changes taking place in the shape of the region occupied by the liquid phase. At any time $t, u(t)$ is the phase interface. The penultimate condition in (4.1) is the mathematical expression for the equation of thermal balance at the interface.

Using the above results, we can formulate the Stefan problem as an optimal control problem for the shape of the region for the heat-conduction equation.

Based on the form of the function $\varphi_0(\cdot)$, one can derive an a priori estimate /5, 6/ for the value of the derivative $u^{\cdot}(t) \equiv v(t)$ in the Stefan problem. We shall assume that this estimate is known: $v(t) \in [0, V]$ and the number X is so chosen that $u(t) \in (0, X)\ \forall t \in [0, T]$.

Consider the corresponding model problem of optimal control:

$$\begin{gathered} y_\varepsilon^{\cdot}(t, x) - y_\varepsilon''(t, x) + \varepsilon^{-1} U_\varepsilon(t, x)\, y_\varepsilon(t, x) = 0, \quad (t, x) \in D \\ y_\varepsilon(0, x) = \varphi(x) = \begin{cases} \varphi_0(x), & x \in [0, u_0] \\ 0, & x \in (u_0, X] \end{cases} \\ y_\varepsilon(t, 0) = 0, \quad y_\varepsilon(t, X) = 0, \quad \forall t \in [0, T] \\ u^{\cdot}(t) = v(t), \quad u(0) = u_0 \end{gathered} \tag{4.2}$$

$$J(\varepsilon, v(\cdot)) = \int_0^T [y_\varepsilon'(t, u(t)) + kv(t)]^2\, dt \to \inf \tag{4.3}$$

Here $U_\varepsilon(t, x)$ is a function corresponding to the "shape" of the region in which the process is evolving; it is equal to

$$U_\varepsilon(t, x) = \begin{cases} 0, & 0 < x \leqslant u(t) \\ w_\varepsilon(x - u(t) - \varepsilon), & u(t) < x \leqslant u(t) + \varepsilon \\ 1, & u(t) + \varepsilon < x < X \end{cases}$$

$$w_\varepsilon(x) = \int_{-\infty}^{x} \omega_\varepsilon(|\xi|)\, d\xi, \quad \omega_\varepsilon(|\xi|) = \begin{cases} 0, & |\xi| \geqslant \varepsilon \\ C\varepsilon^{-1} \exp(-\varepsilon^2(\varepsilon^2 - |\xi|^2)^{-1}), & |\xi| < \varepsilon \end{cases}$$

The number C is determined by the condition

$$\int_{-\infty}^{0} \omega_\varepsilon(|\xi|)\, d\xi = 1$$

In its structure, $U_\varepsilon(t, x)$ is a smooth regularization of the characteristic function $U(t, x)$. The choice of the smooth function $U_\varepsilon(t, x)$ approximating the characteristic function $U(t, x)$ is dictated by the form of the cost criterion (4.3).

Using the technique employed in Theorem 2, one can show that problem (4.2), (4.3) has a solution $y_\varepsilon(t, x; u_\varepsilon^\circ(\cdot)), v_\varepsilon^\circ(\cdot)$, which converges as $\varepsilon \to 0$ to a solution of the Stefan problem. The solution $y_\varepsilon(t, x; u_\varepsilon^\circ(\cdot))$ converges to $y(t, x)$ in the norm of $L_2(D)$, while $v_\varepsilon^\circ(\cdot)$ converges to $v^\circ(\cdot)$ in the weak topology of $L_2(0, T)$.

Thus, the penalty method proposed in this paper enables one to reduce the solution of the Stefan problem to an optimal control problem for the specially constructed system (4.2), whose solution for sufficiently small $\varepsilon > 0$ differs only slightly from the solution of the Stefan problem.

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